ON A LINEAR TRANSCENDENCE MEASURE FOR THE SOLUTIONS OF A UNIVERSAL DIFFERENTIAL EQUATION AT ALGEBRAIC POINTS

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Journal of Mathematical Analysis and Applications 279 (2003), 684-699.

In this paper the author continues his former work on universal differential equations and on the properties of certain solutions at algebraic points. It has been shown that any continuous function on the real line can be approximated by C^∞-solutions y of one specific algebraic differential equation (ADE) such that for every real algebraic number τ the set

$$\left\{ y^{(k)}(\tau)\ (k \geq 1)\ :\ y^{(k)} \cdot y^{(k+1)}(\tau) \neq 0 \right\}$$

is linearly independent over the field of real algebraic numbers, which means that every finite subset is linearly independent (*Journal of Mathematical Analysis and Applications* **256** (2001), 324-338.):

Theorem 1 *There is a nontrivial fifth-order ADE of the form*

$$P(y', y'', \dots, y^{(5)}) = 0 ,$$

with a homogeneous differential polynomial P in five variables with integer coefficients, satisfying the following conditions:
Let f be a real-valued continuous function defined on the real line, and let $(\varepsilon_n)_{n\geq 1}$ be a sequence of positive numbers tending to zero. Then there is a sequence $(y_n)_{n\geq 1}$ of $C^\infty(\mathbb{R})$-solutions of the above ADE such that the functions y_n have the following properties:

$$|f(x) - y_n(x)| < \varepsilon_n \qquad (n \geq 1 ,\ x \in \mathbb{R}) .$$

(i) *Let $(q_k)_{k\geq 1}$ be any sequence of rational numbers. Then, for every integer $n \geq 1$, the set*

$$\left\{ y_n^{(k)}(q_k)\ (k \geq 1)\ :\ y_n^{(k)} \cdot y_n^{(k+1)}(q_k) \neq 0 \right\}$$

is linearly independent over the field of real algebraic numbers.
For every real algebraic number τ the set

$$\left\{ y_n^{(k)}(\tau)\ (k \geq 1)\ :\ y_n^{(k)} \cdot y_n^{(k+1)}(\tau) \neq 0 \right\}$$

is also linearly independent over the field of real algebraic numbers.
(ii) *Let $(\tau_n)_{n\geq 1}$ be any sequence of real algebraic numbers. Then, for every integer $k \geq 0$, the set*

$$\left\{ y_n^{(k)}(\tau_n)\ (n \geq 1)\ :\ y_n^{(k)}(\tau_n) \neq 0 \right\}$$

is linearly independent over the field of real algebraic numbers.
(iii) *Let $n \geq 1$ and $k \geq 0$ be arbitrary integers. Then, for every real algebraic number τ, the number*

$$y_n^{(k)}(\tau)$$

is transcendental, provided that it does not vanish.

In this paper we proceed from this qualitative result to a quantitative version: Let

$$L \;:=\; \sum_{\rho=1}^{r} \psi_\rho \cdot y^{(k_\rho)}(\tau) \tag{1}$$

denote a linear form with algebraic numbers τ, $\psi_1, \ldots, \psi_r$, and integers

$$1 \leq k_1 < k_2 < \ \ldots\ < k_r \;:=\; K\ .$$

In Theorem 2 below a lower bound for $|L|$ is explicitly given, which is called a linear transcendence measure of $y^{(k_1)}(\tau), \ldots, y^{(k_r)}(\tau)$. Apart from some minor arguments the construction of the linear transcendence measure is based on a specific polynomial into which the linear form L can be transformed.
In order to state the theorem we shall need a number of initial parameters. In particular, for any integer z let $J_z := [z; 1+z]$ denote a compact interval of length 1. For any real-valued continuous function f defined on J_z the map given by

$$\omega(f, z; \delta) \;:=\; \sup\{\, |f(x) - f(y)| \;:\; x, y \in J_z \,\wedge\, |x-y| < \delta\}$$

is called the modulus of continuity of the function f on J_z. In what follows f always denotes a continuous function defined on the whole real line. For any integer z and any $\varepsilon > 0$ let m denote the smallest even positive integer such that the inequality

$$\omega\Big(f, z; \frac{1}{m-1}\Big) \;<\; \frac{\varepsilon}{16} \tag{2}$$

holds; such an integer exists since $\omega(f, z; \delta)$ tends to zero with $\delta \to 0^+$ for any continuous function f.
The real number τ is now assumed to be algebraic. Then it satisfies an algebraic identity, say

$$\sum_{\nu=0}^{n_1} b_\nu \tau^\nu \;=\; 0\ ,$$

with integers b_ν, $b := b_{n_1} > 0$. Let

$$C_1 \;:=\; n_1(1+|\tau|)^{n_1-1} \sum_{\nu=1}^{n_1} |b_\nu|\ ,$$

and

$$M := \max\{ 2m ; 1 + 2m|\tau| \} ,$$

where m is the integer satisfying (2) with $\tau \in J_z$. Moreover, the positive integers $a_1, a_2,\ldots,a_r$ are assumed to be the leading coefficients of the minimal polynomials of the algebraic numbers $\psi_1, \psi_2,\ldots,\psi_r$ in L. There is no loss of generality to assume $\psi_r \neq 0$ in (1). We shall also need

$$y^{(K)}y^{(K+1)}(\tau) \neq 0 . \tag{3}$$

If $P(x) = \sum_{\nu=0}^{n} b_\nu x^\nu$ denotes some polynomial with integer coefficients, the height of this polynomial is defined by $H(P) := \max_\nu |b_\nu|$. Moreover, when $P(x)$ is the minimal polynomial of some number α (i.e. $P(\alpha) = 0$, P irreducible over $\mathbb{Q}$ with coprime integers $b_0, \ldots, b_{n_1}$, $b_{n_1} > 0$), the height $h(\alpha)$ of α is given by the height of its minimal polynomial: $h(\alpha) := H(P)$.

Theorem 2 *Let $(\varepsilon_n)_{n\geq 1}$ be the sequence from Theorem 1, let f be any real-valued continuous function defined on the real line, and let $k_1, k_2, \ldots, k_r$ be integers with $1 \leq k_1 < k_2 < \ldots < k_r = K$. Let τ be some real algebraic number of degree n_1 and height h_1, let $\psi_1, \ldots, \psi_r (\neq 0)$ be algebraic numbers with deg$(\psi_\rho) \leq n_2$ and $h(\psi_\rho) \leq h_2$ $(1 \leq \rho \leq r)$. If (2) is assumed for any $\varepsilon = \varepsilon_n$, and when (3) holds for the corresponding approximating function $y = y_n$ from Theorem 1, we have:*

$$| f(x) - y(x) | < \varepsilon \qquad (x \in \mathbb{R}) ,$$

and

$$|L| > \frac{\varepsilon C_2 |\psi_r| m^K (M h_1 h_2)^{-C_3}}{C_1^{K-1}(2m)^{n_1(K-1)} 2^{C_1(2m)^{n_1}}} ,$$

where

$$C_1 = C_1(\tau) = n_1(1+|\tau|)^{n_1-1} \sum_{\nu=1}^{n_1} |b_\nu| , \quad \text{and} \quad M := \max\{ 2m ; 1 + 2m|\tau| \} .$$

The positive numbers C_2 and C_3 depend at most on n_1, n_2, r, and K, and they are effectively computable.

We point out that m and M depend on the modulus of continuity of the function f (restricted on the interval J_z containing τ). Thus the above transcendence measure depends on the behavior of the function f within a specific neighborhood of τ.
A much simpler but less general linear transcendence measure is given by the following corollary. We have $|\tau| \leq 1 + h_1 \leq 2h_1$, such that one gets $M \leq (1 + 4m)h_1$. Therefore it is clear that

$$M h_1 h_2 \leq (1 + 4m) h_1^2 h_2^2$$

holds. Moreover, we now make use of the fact that m depends at most on f and ε, which follows from (2). Altogether, we get:

Corollary 1 *Let $(\varepsilon_n)_{n\geq 1}$ be the sequence from Theorem 1, let f be any real-valued continuous function defined on the real line, and let $k_1, k_2, \ldots, k_r$ be integers with $1 \leq k_1 < k_2 < \ldots < k_r = K$. Let τ be some real algebraic number of degree n_1 and height h_1, let $\psi_1, \ldots, \psi_r (\neq 0)$ be algebraic numbers with deg$(\psi_\rho) \leq n_2$ and $h(\psi_\rho) \leq h_2$ $(1 \leq \rho \leq r)$. If (2) is assumed for* *any $\varepsilon = \varepsilon_n$, and when (3) holds for the corresponding approximating function $y = y_n$ from Theorem 1, we have:*

$$|\, f(x) - y(x)\,| \,<\, \varepsilon \qquad (x \in \mathbb{R}\,)\ ,$$

and

$$|L| \,>\, \frac{C_5|\psi_r|(h_1h_2)^{-C_4}}{C_1^{K-1}C_6^{C_1}}\ ,$$

where

$$C_1 = C_1(\tau) \,:=\, n_1(1+|\tau|)^{n_1-1}\sum_{\nu=1}^{n_1}|b_\nu|\ .$$

Each of the three positive numbers

$$C_4 = 2C_3 = C_4(n_1, n_2, r, K)\,, \quad C_5 = C_5(n_1, n_2, r, K; f, \varepsilon)\,, \quad C_6 = C_6(n_1; f, \varepsilon)$$

is effectively computable.

From Theorem 1 we know that all the five derivatives $y'(\tau), \ldots, y^{(5)}(\tau)$ satisfy a nonlinear algebraic differential equation with integer coefficients.

Theorem 3 *A specific differential equation in Theorem 1 is the following one. It is homogeneous of degree 11 with 17 terms of weight 22. The corresponding differential polynomial is irreducible over* $\mathbb{Q}$*:*

$$\begin{gathered}
3y'^6y''^3{y^{(5)}}^2 + 8y'^6y''^2{y^{(4)}}^3 - 24y'^6y''^2y'''y^{(4)}y^{(5)} + 12y'^6y''y'''^2{y^{(4)}}^2 + 18y'^6y''y'''^3y^{(5)} \\
-18y'^6y'''^4y^{(4)} + 12y'^4y''^5{y^{(4)}}^2 + 6y'^4y''^5y'''y^{(5)} - 60y'^4y''^4y'''^2y^{(4)} + 45y'^4y''^3y'''^4 \\
-6y'^3y''^7y^{(5)} + 24y'^3y''^6y'''y^{(4)} - 18y'^3y''^5y'''^3 + 6y'^2y''^8y^{(4)} - 6y'^2y''^7y'''^2 \\
-6y'y''^9y''' + 4y''^{11} \,=\, 0\ .
\end{gathered}$$