

# ON THE APPROXIMATION OF CONTINUOUS FUNCTIONS BY ANALYTIC SOLUTIONS OF UNIVERSAL FUNCTIONAL-DIFFERENTIAL EQUATIONS

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We first consider continuous functions  $f$  on  $\mathbb{R}$  where both the limits

$$\lim_{x \rightarrow -\infty} f(x) \quad \text{and} \quad \lim_{x \rightarrow +\infty} f(x) \quad (1)$$

exist in  $\mathbb{R}$ . One gets:

**Theorem 1** *There exists a nontrivial sixth-order ADE of the form*

$$P(x; y', y'', \dots, y^{(6)}) = 0 \quad (2)$$

*such that any real continuous function  $f$  on the real line can be uniformly approximated by the real analytic solutions of (2), provided that the limits in (1) exist in  $\mathbb{R}$ .*

**Corollary 1** *There exists a nontrivial ADE of the form*

$$P(y', y'', \dots, y^{(7)}) = 0$$

*such that the real-analytic solutions have the same property as the real-analytic solutions of the ADE (2).*

**Theorem 2** *There exists a nontrivial AFE of the form*

$$P(y'(x), y'(x + \log 2), \dots, y'(x + 5 \log 2)) = 0 \quad (3)$$

*such that for any compact interval  $I$  the real-analytic solutions of (3) (defined on the whole real line) are dense in  $C(I)$ .*

**Theorem 3** *Let  $n$  denote some positive integer, and let  $P$  be some non-zero polynomial having real coefficients. Then, for every positive real number  $\Delta$  and any compact interval  $I \subset \mathbb{R}$  satisfying  $|I| > 2n\Delta$ , the continuous solutions  $g$  of the AFE*

$$P(g(x), g(x + \Delta), \dots, g(x + n\Delta)) = 0 \quad (x \pm n\Delta \in I)$$

*are not dense in  $C(I)$ .*

By  $M(\mathbf{C})$  we denote the set of meromorphic functions  $f$  defined on  $\mathbf{C}$  such that additionally two conditions are satisfied:

- (i)  $f(z)$  is analytically at every point  $z$  with  $\Im z = 0, \pm 2\pi, \pm 4\pi$ .
- (ii)  $f(z)$  takes real values at points  $z$  from the real axis.

Particularly every function  $f(z)$  from  $M(\mathbf{C})$ , where  $z$  is restricted on the real line, represents a real-valued analytical function from  $C^\omega(\mathbf{R})$ , and there is some real-valued analytical antiderivative defined on the real axis.

**Theorem 4** *Let  $I$  denote a compact interval. Then every function from  $C(I)$  can be uniformly approximated by real-valued analytic functions  $y$  defined on  $I$ , such that  $y' \in M(\mathbf{C})$  holds and  $y'$  satisfies the AFE*

$$\begin{aligned} & y_1 y_2^2 y_3 y_5 - y_1 y_2^2 y_4^2 - y_1 y_2^2 y_4 y_5 + y_1 y_2 y_3 y_4^2 + y_1 y_2 y_4^2 y_5 \\ & - y_1 y_3 y_4^2 y_5 - y_2^2 y_3 y_4 y_5 + y_2^2 y_4^2 y_5 = 0 \end{aligned} \quad (4)$$

with

$$\begin{aligned} y_1 &:= y'(x - 4\pi i), \quad y_2 := y'(x - 2\pi i), \quad y_3 := y'(x), \\ y_4 &:= y'(x + 2\pi i), \quad y_5 := y'(x + 4\pi i). \end{aligned}$$

The identity in (4) represents an AFE of order one for real-valued analytical functions on arbitrary compact intervals  $I$ .