ON THE APPROXIMATION OF REAL CONTINUOUS FUNCTIONS BY SERIES OF SOLUTIONS OF A SINGLE SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

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Colloquium Mathematicum 104 no.1 (2006), 57-84.

Let $\omega \in \mathbb{R}^s \to \mathbb{R}_{>0}$ denote some bounded continuous weight function taking positive values everywhere, such that additionally

$$\int_{\mathbb{R}^s} \omega(x_1, \ldots, x_s)\, d\mathbf{x} \;=\; 1$$

holds. For the sake of brevity we use the notation $d\mathbf{x}$ to express $dx_1 dx_2 \ldots dx_s$. Now let

$$\|g\|_\omega \;:=\; \int_{\mathbb{R}^s} \omega(x_1, \ldots, x_s)|g(x_1, \ldots, x_s)|\, d\mathbf{x} \qquad (g \in C(\mathbb{R}^s)) \;.$$

Theorem 1 *There exists an effectively computable polynomial* $P(x, t_0, \ldots, t_5) \in \mathbb{Z}[x, t_0, \ldots, t_5]$ *having the following property.*
Let $s \geq 1$ *be some integer, let* $f \in \mathbb{R}^s \to \mathbb{R}$ *be some continuous function defined on* $\mathbb{R}^s$ *and let* ε *be some arbitrary positive number. Then a series* $H \in C^\infty(\mathbb{R}^s)$ *of analytic functions* $H_r \in C^\omega(\mathbb{R}^s)$ *exists, say*

$$H(x_1, \ldots, x_s) \;=\; \sum_{r=1}^{\infty} H_r(x_1, \ldots, x_s) \qquad (x_\nu \in \mathbb{R}\;;\; \nu = 1, \ldots, s)\;,$$

such that $\|f - H\|_\omega < \varepsilon$ *holds, and every analytic function* H_r *solves the system of partial differential equations*

$$P\Big(x_\sigma\,;\, H_r\,,\, \frac{\delta H_r}{\delta x_\sigma}\,,\, \ldots\,,\, \frac{\delta^5 H_r}{\delta x_\sigma^5}\Big) \;=\; 0 \qquad (\sigma = 1, \ldots, s)\;. \tag{1}$$

A specific polynomial $P(x, t_0, \ldots, t_5)$ *is homogeneous of degree 16 in its variables* $t_0, \ldots, t_5$, *and it consists of 575 terms of the form*

$$a \cdot x^b \cdot t_0^{c_0} \cdot \ldots \cdot t_5^{c_5} \qquad (a, b, c_0, \ldots, c_5 \in \mathbb{Z}\;;\; b, c_0, \ldots, c_5 \geq 0\;;\; c_0 + \ldots + c_5 = 16)\;.$$

Using standard arguments it follows easily from (1) that H_r also satisfies a system of autonomous partial differential equations of order six.