

# ON A SEQUENCE TRANSFORMATION WITH INTEGRAL COEFFICIENTS FOR EULER'S CONSTANT, II

Carsten Elsner

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Let

$$s_1 := 0, \quad s_n = 1 + 1/2 + \dots + 1/(n-1) - \log n \quad (n \geq 2).$$

In 1995, the author <sup>1</sup> has found a series transformation of the type  $\sum_{k=0}^n \mu_{n,k,\tau} s_{k+\tau}$  with integer coefficients  $\mu_{n,k,\tau}$ , from which geometric convergence to Euler's constant  $\gamma$  for  $\tau = \mathcal{O}(n)$  results. In recently published papers *T.Rivoal* and *Kh. and T.Hessami Pilehrood* have generalized this result. In the present paper we introduce a series transformation  $\sum_{k=0}^n \mu_{n,k,\tau_1} s_{k+\tau_2}$  with two parameters  $\tau_1$  and  $\tau_2$  satisfying  $\tau_1 + 1 \leq \tau_2 \leq n + \tau_1 + 1$ , and integer coefficients  $\mu_{n,k,\tau_1}$ . By applying the analysis of the  $\psi$ -function, we prove a sharp bound for  $|S - \gamma|$ . A similar result holds for *generalized Stieltjes constants*. Let  $a \geq 0$  be a real number, and let

$$s_1(a) := 0, \quad s_n(a) := \left( \frac{1}{a+1} + \frac{1}{a+2} + \frac{1}{a+3} + \dots + \frac{1}{a+n-1} \right) - \log(a+n) \quad (n \geq 2).$$

The sequence  $(s_n(a))_{n \geq 1}$  is convergent for any real number  $a > 0$ :

$$\lim_{n \rightarrow \infty} s_n(a) = \gamma_0(a) - \frac{1}{a},$$

where  $\gamma_0(a)$  are known as *generalized Stieltjes constants of order 0*, i.e.

$$\gamma_0(a) := -\frac{\Gamma'(a)}{\Gamma(a)} = -\Psi(a).$$

Particularly, we have for  $a = 0$ :

$$\lim_{n \rightarrow \infty} s_n(0) = \lim_{a \rightarrow 0} \left( \gamma_0(a) - \frac{1}{a} \right) = \gamma.$$

Our main results are given by the following theorems:

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<sup>1</sup>*On a sequence transformation with integral coefficients for Euler's constant*, Proceedings of the AMS **123** no.5 (1995), 1537-1541

**Theorem 1** Let  $n \geq 1$ ,  $\tau_1 \geq 1$  and  $\tau_2 \geq 1$  be integers. Additionally we assume that

$$1 + \tau_1 \leq \tau_2 .$$

Then one has

$$\begin{aligned} & \left| \sum_{k=0}^n (-1)^{n+k} \binom{n + \tau_1 + k}{n} \binom{n}{k} \cdot s_{k+\tau_2}(a) - \left( \gamma_0(a) - \frac{1}{a} \right) \right| \\ &= \int_0^1 \left( \frac{1}{1-u} + \frac{1}{\log u} \right) \cdot u^{a+\tau_2-\tau_1-1} \cdot \frac{d^n}{du^n} \left( \frac{u^{n+\tau_1}(1-u)^n}{n!} \right) du . \end{aligned}$$

**Theorem 2** Let  $n \geq 1$ ,  $\tau_1 \geq 1$  and  $\tau_2 \geq 1$  be integers. Additionally we assume that

$$1 + \tau_1 \leq \tau_2 \leq 1 + n + \tau_1 .$$

Then one has

$$\begin{aligned} & \left| \sum_{k=0}^n (-1)^{n+k} \binom{n + \tau_1 + k}{n} \binom{n}{k} \cdot s_{k+\tau_2}(0) - \gamma \right| \\ &= \int_0^1 \int_0^1 w(t) \cdot \frac{(1-u)^{n+\tau_1} u^n (1-t)^{\tau_2-\tau_1-1} t^{n+\tau_1-\tau_2+1}}{(1-ut)^{n+1}} du dt , \end{aligned}$$

with

$$w(t) := \frac{1}{t \cdot \left( \pi^2 + \log^2\left(\frac{1}{t} - 1\right) \right)} .$$

Setting

$$n = \tau_2 = dm , \quad \tau_1 = (d-1)m - 1 \quad (d \geq 2) ,$$

we get an explicit upper bound from Theorem 2:

**Corollary 1** For integers  $m \geq 2$ ,  $d \geq 3$ , we have

$$\left| \sum_{k=0}^{dm} (-1)^{dm+k} \binom{(2d-1)m+k-1}{dm} \binom{dm}{k} \cdot s_{k+dm}(0) - \gamma \right| < C_d \cdot \left( \frac{\left(1 - \frac{1}{d}\right)^d}{(d-1)4^d} \right)^{m-2} ,$$

where  $0 < C_d \leq 1/16\pi^2$  is some constant depending only on  $d$ . For  $d = 2$  one gets

$$\left| \sum_{k=0}^{2m} (-1)^k \binom{3m+k-1}{2m} \binom{2m}{k} \cdot s_{k+2m}(0) - \gamma \right| < \left( \frac{16}{7\pi} \right)^2 \cdot \frac{1}{64^m} \quad (m \geq 1) .$$

Our method can be modified to work in general for series transformations connected with Ser-type formulas like

$$\gamma - s_n(0) = \sum_{m=1}^{\infty} \frac{\Gamma(m)}{\binom{n}{m}} t_{m+1} \quad \text{and} \quad \gamma_0(a) - s_n(a) - \frac{1}{a} = \sum_{m=1}^{\infty} \frac{\Gamma(m)}{(a+n)_m} t_{m+1} \quad (a > 0)$$

with rational numbers  $t_{m+1}$  defined by

$$t_{m+1} := -\frac{1}{m!} \cdot \int_0^1 (-x)(1-x)(2-x) \cdots (m-1-x) dx \quad (m \geq 1).$$

Then, in the case of  $\gamma$ , we apply the Mellin - Barnes integral representation of the  ${}_3F_2$  - function, and use different combinatorial identities. Following this different approach, which is more technically than the method used below, we get somewhat weaker results than that ones stated in Theorem 2 and Corollary 1:

Let  $n \geq 1$ ,  $\tau_1 \geq 1$  and  $\tau_2 \geq 1$  be integers. Additionally we assume that

$$\tau_1 + 1 \leq \tau_2 \leq n + \tau_1 + 1.$$

Then one has

$$\begin{aligned} & \left| \sum_{k=0}^n (-1)^{n+k} \binom{n+\tau_1+k}{n} \binom{n}{k} \cdot s_{k+\tau_2} - \gamma \right| \\ & \leq \frac{(\tau_2-1)!(n+\tau_1)!(n+\tau_1-\tau_2+1)!}{2(n+\tau_1-\tau_2+2)(2n+\tau_1+1)!\tau_1!} \cdot \max_{0 \leq x \leq 1} \left| \sum_{l=0}^{\tau_2-\tau_1-1} \frac{(\tau_1-\tau_2+1)_l (n+\tau_1+1)_l}{l!(\tau_1+1)_l} \right. \\ & \quad \left. \times {}_3F_2 \left( \begin{matrix} n+\tau_1-\tau_2+2 & n+\tau_1-\tau_2+2-x & n-l+1 \\ 2n+\tau_1+2 & n+\tau_1-\tau_2+3 & 1 \end{matrix} \right) \right| \end{aligned}$$

For  $n = \tau_2 = 2m$ ,  $\tau_1 = m - 1$  we then have for every integer  $m \geq 1$  that

$$\begin{aligned} & \left| \sum_{k=0}^{2m} (-1)^k \binom{3m+k-1}{2m} \binom{2m}{k} s_{k+2m} - \gamma \right| \\ & \leq \frac{m}{2} \cdot |\zeta(2) - q_m| = \frac{m}{2} \cdot \int_0^1 \int_0^1 \frac{z^{m-1} w^{2m} (1-z)^{m+1} (1-w)^{3m-1}}{(1-wz)^{2m+1}} dw dz \leq \frac{2m}{64^m}, \end{aligned}$$

where

$$q_m := \sum_{k=0}^{2m} (-1)^k \binom{3m+k-1}{2m} \binom{2m}{k} \cdot \sum_{\nu=1}^{k+2m} \frac{1}{\nu^2}.$$

Here a Beuker's type integral for  $\zeta(2)$  pops up.