ALGEBRAIC RELATIONS FOR RECIPROCAL SUMS OF EVEN TERMS IN FIBONACCI NUMBERS

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In this paper, we discuss the algebraic independence and algebraic relations for reciprocal sums of even terms in Fibonacci numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}^{2s}} \, ,$$

and for sums of evenly even and unevenly even types

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n}^{2s}} \, , \qquad \sum_{n=1}^{\infty} \frac{1}{F_{4n-2}^{2s}}$$

as well. We prove that the numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n-2}^{2}} \, , \qquad \sum_{n=1}^{\infty} \frac{1}{F_{4n-2}^{4}} \, , \qquad \sum_{n=1}^{\infty} \frac{1}{F_{4n-2}^{6}}$$

are algebraically independent, and write each

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n-2}^{2s}} \qquad (s \geq 4)$$

as an explicit rational function of these three numbers over $\mathbb{Q}$. Similar results are obtained for various series of even type including the reciprocal sums of Lucas numbers

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n}^{p}} \, , \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n}^{p}} \, , \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n-2}^{p}} \, .$$

Key words: Algebraic independence, Fibonacci numbers, Lucas numbers, Jacobian elliptic functions, Ramanujan functions, q-series, Nesterenko's theorem.