ON THE RESIDUE CLASSES OF INTEGER SEQUENCES SATISFYING A LINEAR THREE TERM RECURRENCE FORMULA

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In this paper we investigate linear three-term recurrence formulae $Z_n = T(n)Z_{n-1}+U(n)Z_{n-2}$ $(n \geq 2)$ with sequences of integers $(T(n))_{n\geq 0}$ and $(U(n))_{n\geq 0}$, which are ultimately periodic modulo m, e.g.

$$\begin{aligned} (T(n) \mod m)_{n\geq 0} &= \left(a_0, a_1, a_2, \ldots, a_\rho, \overline{T_1, T_2, \ldots, T_w} \right) , \\ (U(n) \mod m)_{n\geq 0} &= \left(b_0, b_1, b_2, \ldots, b_\rho, \overline{U_1, U_2, \ldots, U_w} \right) . \end{aligned}$$

In a former paper the authors computed explicitly the coefficients of a linear three-term recurrence formula for $z_n = Z_{rn+i}$ with $0 \leq i < r$, when $(T(n))_{n\geq 0}$ and $(U(n))_{n\geq 0}$ belong to regular or non-regular Hurwitz-type continued fraction expansions. Using this result we show now that the sequence $(Z_n)_{n\geq 0}$ is ultimately periodic modulo m. As a consequence, for Hurwitz-type continued fraction expansions $\alpha = [a_0; \overline{T_1(k), \ldots, T_r(k)}]_{k=1}^{\infty}$ or $\alpha = [a_0; a_1, \overline{T_1(k), \ldots, T_r(k)}]_{k=1}^{\infty}$ with polynomials $T_1 \neq const., T_2, \ldots, T_r$ we deduce for all positive integers a and m that $\liminf_q q\|q\alpha\| = 0$, where $q \equiv a \mod m$ and $\|.\|$ denotes the distance from the nearest integer. Finally, we are particularly interested in the recurrence formula $Z_n = (an + b)Z_{n-1} + cZ_{n-2}$ $(n \geq 2)$ and compute the length of a period of the sequence $(Z_n \mod m)_{n\geq 0}$, when (a, m) divides b, and $(c, m) = 1$. This generalizes former results of the authors dealing with regular continued fraction expansions of the numbers $\exp(1/s)$ for integers $s \geq 1$.

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