ALGEBRAIC INDEPENDENCE RESULTS FOR RECIPROCAL SUMS OF FIBONACCI NUMBERS

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(Acta Arithmetica 148.3 (2011), 205–223)

Let $(U_n)_{n\geq 0}$ be a sequence of numbers given by $U_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ for $n \geq 0$, where α and β are complex numbers satisfying $|\beta| < 1$ and $\alpha\beta = -1$. Let $s \geq 1$ and

$$\Phi_{2s} := (\alpha - \beta)^{-2s} \sum_{n=1}^{\infty} \frac{1}{U_n^{2s}} \ .$$

Recently, the authors have shown that in the case of algebraic α, β the reciprocal sums Φ_2, Φ_4, Φ_6 are algebraically independent over $\mathbb{Q}$ and that for any $s > 3$ every Φ_{2s} can be expressed explicitly as an algebraic function of Φ_2, Φ_4, and Φ_6. In this paper we prove that for any distinct positive integers s_1, s_2, s_3 corresponding to algebraic α, β with $|\beta| < 1$ and $\alpha\beta = -1$ the numbers Φ_{2s_1}, Φ_{2s_2} and Φ_{2s_3} are algebraically independent over $\mathbb{Q}$ if and only if at least one of s_1, s_2, s_3 is even. In particular, for $\beta = (1-\sqrt{5})/2$ and $\beta = 1 - \sqrt{2}$, our results on Φ_{2s} are applicable to reciprocal sums on Fibonacci numbers $U_n = F_n$ and on Pell numbers $U_n = P_n$, respectively.

MR Subject Classification Numbers: Primary 11J81, Secondary 11J91, 33E05, 11R04