ON A UNIVERSAL DIFFERENTIAL EQUATION
FOR THE ANALYTIC TERMS OF
$C^\infty$-SUPERPOSITIONS ON THE REAL LINE

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Let $\omega \in C(\mathbb{R}) \to \mathbb{R}_{>0}$ denote a bounded continuous weight function taking positive values everywhere, such that
\[ \int_{-\infty}^{\infty} \omega(x) \, dx = 1 \]
holds. Then, by
\[ \|g\|_\omega := \int_{-\infty}^{\infty} \omega(x) |g(x)| \, dx \quad (g \in C(\mathbb{R})) , \]
a norm for continuous functions $g$ on the real line is given.

**Theorem 1** There exists a nontrivial autonomous algebraic differential equation $P = 0$ of order at most 7, where $P$ denotes an effectively computable polynomial in at most eight variables, having the following property.

Let $f \in C(\mathbb{R}) \to \mathbb{R}$ be some continuous function defined on the real line and let $\varepsilon$ be some arbitrary positive number. Then a superposition $H \in C^\infty(\mathbb{R})$ of analytic functions $H_r \in C^\omega(\mathbb{R})$ exists, say
\[ H(x) = \sum_{-\infty < r < \infty} H_r(x) \quad (x \in \mathbb{R}) , \]
such that $\|f - H\|_\omega < \varepsilon$ holds, and every analytic function $H_r$ solves the above universal differential equation. Moreover, every analytic function $H_r$ on $\mathbb{R}$ is an entire function on $\mathbb{C}$. 