ON THE APPROXIMATION OF REAL CONTINUOUS FUNCTIONS BY SERIES OF SOLUTIONS OF A SINGLE SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

Carsten Elsner

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Let \( \omega \in \mathbb{R}^s \rightarrow \mathbb{R}_{>0} \) denote some bounded continuous weight function taking positive values everywhere, such that additionally

\[
\int_{\mathbb{R}^s} \omega(x_1, \ldots, x_s) \, dx = 1
\]

holds. For the sake of brevity we use the notation \( dx \) to express \( dx_1 dx_2 \ldots dx_s \). Now let

\[
\|g\|_\omega := \int_{\mathbb{R}^s} \omega(x_1, \ldots, x_s) |g(x_1, \ldots, x_s)| \, dx \quad (g \in C(\mathbb{R}^s)).
\]

Theorem 1

There exists an effectively computable polynomial \( P(x, t_0, \ldots, t_5) \in \mathbb{Z}[x, t_0, \ldots, t_5] \) having the following property. Let \( s \geq 1 \) be some integer, let \( f \in \mathbb{R}^s \rightarrow \mathbb{R} \) be some continuous function defined on \( \mathbb{R}^s \) and let \( \varepsilon \) be some arbitrary positive number. Then a series \( H \in C^\infty(\mathbb{R}^s) \) of analytic functions \( H_r \in C^\omega(\mathbb{R}^s) \) exists, say

\[
H(x_1, \ldots, x_s) = \sum_{r=1}^{\infty} H_r(x_1, \ldots, x_s) \quad (x_r \in \mathbb{R} ; \, \nu = 1, \ldots, s),
\]

such that \( \|f - H\|_\omega < \varepsilon \) holds, and every analytic function \( H_r \) solves the system of partial differential equations

\[
P\left(x_\sigma ; H_r, \frac{\delta H_r}{\delta x_\sigma}, \ldots, \frac{\delta^5 H_r}{\delta x_\sigma^5}\right) = 0 \quad (\sigma = 1, \ldots, s).
\]

A specific polynomial \( P(x, t_0, \ldots, t_5) \) is homogeneous of degree 16 in its variables \( t_0, \ldots, t_5 \), and it consists of 575 terms of the form

\[
a \cdot x^b \cdot t_0^{c_0} \cdot \ldots \cdot t_5^{c_5} \quad (a, b, c_0, \ldots, c_5 \in \mathbb{Z} ; \, b, c_0, \ldots, c_5 \geq 0 ; \, c_0 + \ldots + c_5 = 16).
\]

Using standard arguments it follows easily from (1) that \( H_r \) also satisfies a system of autonomous partial differential equations of order six.