ON A SEQUENCE TRANSFORMATION WITH INTEGRAL COEFFICIENTS FOR EULER’S CONSTANT, II

Carsten Elsner

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Let

\[ s_1 := 0, \quad s_n = 1 + 1/2 + \ldots + 1/(n - 1) - \log n \quad (n \geq 2). \]

In 1995, the author \(^1\) has found a series transformation of the type \( \sum_{k=0}^{n} \mu_{n,k,\tau} s_k + \tau \) with integer coefficients \( \mu_{n,k,\tau} \), from which geometric convergence to Euler’s constant \( \gamma \) for \( \tau = O(n) \) results. In recently published papers T.Rivoal and Kh. and T.Hessami Pilehrood have generalized this result. In the present paper we introduce a series transformation \( \sum_{k=0}^{n} \mu_{n,k,\tau_1,\tau_2} s_k + \tau_2 \) with two parameters \( \tau_1 \) and \( \tau_2 \) satisfying \( \tau_1 + 1 \leq \tau_2 \leq n + \tau_1 + 1 \), and integer coefficients \( \mu_{n,k,\tau_1} \). By applying the analysis of the \( \psi \) - function, we prove a sharp bound for \( |S - \gamma| \). A similar result holds for generalized Stieltjes constants. Let \( a \geq 0 \) be a real number, and let

\[ s_1(a) := 0, \quad s_n(a) := \left( \frac{1}{a+1} + \frac{1}{a+2} + \frac{1}{a+3} + \ldots + \frac{1}{a+n-1} \right) - \log(a+n) \quad (n \geq 2). \]

The sequence \( (s_n(a))_{n \geq 1} \) is convergent for any real number \( a > 0 \):

\[ \lim_{n \to \infty} s_n(a) = \gamma_0(a) - \frac{1}{a}, \]

where \( \gamma_0(a) \) are known as generalized Stieltjes constants of order 0, i.e.

\[ \gamma_0(a) := \frac{\Gamma'(a)}{\Gamma(a)} = -\Psi(a). \]

Particularly, we have for \( a = 0 \):

\[ \lim_{n \to \infty} s_n(0) = \lim_{a \to 0} \left( \gamma_0(a) - \frac{1}{a} \right) = \gamma. \]

Our main results are given by the following theorems:

\(^1\) On a sequence transformation with integral coefficients for Euler’s constant, Proceedings of the AMS 123 no.5 (1995), 1537-1541
Theorem 1 Let \( n \geq 1, \tau_1 \geq 1 \) and \( \tau_2 \geq 1 \) be integers. Additionally we assume that 
\[ 1 + \tau_1 \leq \tau_2 . \]
Then one has 
\[
\left| \sum_{k=0}^{n} (-1)^{n+k} \binom{n + \tau_1 + k}{n} \binom{n}{k} \cdot s_{k+\tau_2}(a) - \left( \gamma_0(a) - \frac{1}{a} \right) \right| 
= \int_0^1 \left( \frac{1}{1-u} + \frac{1}{\log u} \right) \cdot u^{a+\tau_2-\tau_1-1} \cdot \frac{d^n}{du^n} \left( \frac{u^{n+\tau_1}(1-u)^n}{n!} \right) \, du .
\]

Theorem 2 Let \( n \geq 1, \tau_1 \geq 1 \) and \( \tau_2 \geq 1 \) be integers. Additionally we assume that 
\[ 1 + \tau_1 \leq \tau_2 \leq 1 + n + \tau_1 . \]
Then one has 
\[
\left| \sum_{k=0}^{n} (-1)^{n+k} \binom{n + \tau_1 + k}{n} \binom{n}{k} \cdot s_{k+\tau_2}(0) - \gamma \right| 
= \int_0^1 \int_0^1 w(t) \cdot \frac{(1-u)^{n+\tau_1}u^n(1-t)^{\tau_2-\tau_1-1}t^{n+\tau_1-\tau_2+1}}{(1-ut)^{n+1}} \, du \, dt ,
\]
with 
\[
w(t) := \frac{1}{t \cdot (\pi^2 + \log^2(\frac{1}{t} - 1))} .
\]
Setting 
\[ n = \tau_2 = dm , \quad \tau_1 = (d-1)m - 1 \quad (d \geq 2) , \]
we get an explicit upper bound from Theorem 2:

Corollary 1 For integers \( m \geq 2, d \geq 3 \), we have 
\[
\left| \sum_{k=0}^{dm} (-1)^{dm+k} \binom{(2d-1)m + k - 1}{dm} \binom{dm}{k} \cdot s_{k+dm}(0) - \gamma \right| < C_d \cdot \left( \frac{1 - \frac{1}{d}}{(d-1)4^d} \right)^{m-2} \cdot \frac{1}{64^m} \quad (m \geq 1) .
\]
where \( 0 < C_d \leq 1/16\pi^2 \) is some constant depending only on \( d \). For \( d = 2 \) one gets 
\[
\left| \sum_{k=0}^{2m} (-1)^k \binom{3m + k - 1}{2m} \binom{2m}{k} \cdot s_{k+2m}(0) - \gamma \right| < \left( \frac{16}{7\pi} \right)^2 \cdot \frac{1}{64^m} \quad (m \geq 1) .
\]
Our method can be modified to work in general for series transformations connected with Ser-type formulas like

\[
\gamma - s_n(0) = \sum_{m=1}^{\infty} \frac{\Gamma(m)}{(n)_m} t_{m+1} \quad \text{and} \quad \gamma_0(a) - s_n(a) - \frac{1}{a} = \sum_{m=1}^{\infty} \frac{\Gamma(m)}{(a+n)_m} t_{m+1} \quad (a > 0)
\]

with rational numbers \( t_{m+1} \) defined by

\[
t_{m+1} := -\frac{1}{m!} \cdot \int_0^1 (-x)(1-x)(2-x) \cdots (m-1-x) \, dx \quad (m \geq 1) .
\]

Then, in the case of \( \gamma \), we apply the Mellin - Barnes integral representation of the \( 3F_2 \) -function, and use different combinatorial identities. Following this different approach, which is more technically than the method used below, we get somewhat weaker results than that ones stated in Theorem 2 and Corollary 1:

Let \( n \geq 1, \tau_1 \geq 1 \) and \( \tau_2 \geq 1 \) be integers. Additionally we assume that

\[
\tau_1 + 1 \leq \tau_2 \leq n + \tau_1 + 1 .
\]

Then one has

\[
\left| \sum_{k=0}^{n} (-1)^{n+k} \binom{n + \tau_1 + k}{n} \binom{n}{k} \cdot s_{k+\tau_2} - \gamma \right|
\leq \frac{(\tau_2 - 1)! (n + \tau_1)! (n + \tau_1 - \tau_2 + 1)!}{2(n + \tau_1 - \tau_2 + 2)(2n + \tau_1 + 1)! \tau_1!} \cdot \max_{0 \leq x \leq 1} \left| \sum_{l=0}^{\tau_2 - \tau_1 - 1} \frac{(\tau_1 - \tau_2 + 1)_{l}(n + \tau_1 + 1)_{l}}{l!(\tau_1 + 1)_{l}} \right|
\times {}_3F_2\left( \begin{array}{c} n + \tau_1 - \tau_2 + 2 \\ 2n + \tau_1 + 2 \end{array} \begin{array}{c} n + \tau_1 - \tau_2 + 2 - x \\ n - l + 1 \end{array} \begin{array}{c} n - l + 1 \\ n + \tau_1 - \tau_2 + 3 \end{array} \right) \right|
\]

For \( n = \tau_2 = 2m, \tau_1 = m - 1 \) we then have for every integer \( m \geq 1 \) that

\[
\left| \sum_{k=0}^{2m} (-1)^{k} \binom{3m + k - 1}{2m} \binom{2m}{k} s_{k+2m} - \gamma \right|
\leq \frac{m}{2} \cdot |\zeta(2) - q_m| = \frac{m}{2} \cdot \int_0^1 \int_0^1 \frac{z^{m-1}w^{2m}(1-z)^{m+1}(1-w)^{3m-1}}{(1-wz)^{2m+1}} \, dw \, dz \leq \frac{2m}{64m} ,
\]

where

\[
q_m := \sum_{k=0}^{2m} (-1)^{k} \binom{3m + k - 1}{2m} \binom{2m}{k} \cdot \sum_{\nu=1}^{k+2m} \frac{1}{\nu^2} .
\]

Here a Beuker’s type integral for \( \zeta(2) \) pops up.