EXCEPTIONAL ALGEBRAIC RELATIONS FOR RECIPROCAL SUMS OF FIBONACCI AND LUCAS NUMBERS

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Let \((F_n)_{n\geq0}\) and \((L_n)_{n\geq0}\) be the sequences of Fibonacci and Lucas numbers, respectively. Moreover, we define a set \(\Gamma\) of 12 real numbers by

\[
\Gamma := \left\{ \sum_{n=1}^{\infty} \frac{1}{F_{2n}^s} , \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_{2s}^n} , \sum_{n=1}^{\infty} \frac{1}{F_{2n}^s} , \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{L_{2s}^n} \ (s = 1, 2, 3) \right\} .
\]

From former investigations of the authors it is known that every 4-subset of \(\Gamma\) consisting of four numbers is algebraically dependent over \(\mathbb{Q}\). It is also proven by the authors that each of the 3-subsets

\[
\left\{ \sum_{n=1}^{\infty} \frac{1}{F_{2s}^n} \ (s = 1, 2, 3) \right\} , \left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_{2s}^n} \ (s = 1, 2, 3) \right\} , \left\{ \sum_{n=1}^{\infty} \frac{1}{L_{2s}^n} \ (s = 1, 2, 3) \right\} , \left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{L_{2s}^n} \ (s = 1, 2, 3) \right\}
\]

is algebraically independent over \(\mathbb{Q}\). There are

\[
\binom{12}{3} = 220
\]

3-subsets of \(\Gamma\). In this paper it is shown that among them there are exactly 22 such 3-subsets which are algebraically dependent over \(\mathbb{Q}\). We compute the corresponding polynomials explicitly. For instance, we have

\[
2 \sum_{n=1}^{\infty} \frac{1}{F_{2n}^2} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_{2s}^n} - 5 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{L_{2s}^n} = 0 ,
\]

\[
\left(\sum_{n=1}^{\infty} \frac{1}{F_{2n}^s}\right) \cdot \left(8 \sum_{n=1}^{\infty} \frac{1}{L_{2n}^2} + 1\right) - 5 \sum_{n=1}^{\infty} \frac{1}{L_{2n}^2} - 20 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{L_{2n}^4} = 0 .
\]

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