In its most elaborate form, the Jacobi theta function is defined for two complex variables $z$ and $\tau$ by
\[ \theta(z|\tau) = \sum_{\nu=-\infty}^{\infty} e^{\pi i \nu^2 \tau + 2\pi i \nu z} , \]
which converges for all complex number $z$, and $\tau$ in the upper half-plane. The special case
\[ \theta_3(\tau) := \theta(0|\tau) = 1 + 2 \sum_{\nu=1}^{\infty} e^{\pi i \nu^2 \tau} , \]
is called a Jacobi theta-constant or Thetanullwert of the Jacobi theta function $\theta(z|\tau)$. In this paper, we prove the algebraic independence results for the values of the Jacobi theta-constant $\theta_3(\tau)$. For example, the three values $\theta_3(\tau)$, $\theta_3(n\tau)$, and $D\theta_3(\tau)$ are algebraically independent over $\mathbb{Q}$ for any $\tau$ such that $q = e^{i\pi \tau}$ is an algebraic number, where $n \geq 2$ is an integer and $D := (\pi i)^{-1} d/d\tau$ is a differential operator. This generalizes a result of the first author, who proved the algebraic independence of the two values $\theta_3(\tau)$ and $\theta_3(n\tau)$ for $m \geq 1$. As an application of our main theorem, the algebraic dependence over $\mathbb{Q}$ of the three values $\theta_3(\ell\tau)$, $\theta_3(m\tau)$, and $\theta_3(n\tau)$ for integers $\ell, m, n \geq 1$ is also presented.

2010 MS Classification numbers: 11J85, 11J91, 11F27  
Key Words: Algebraic independence, Theta-constants, Nesterenko’s theorem, Modular equations